

# Seesaw words in Thompson's group $F$

Sean Cleary and Jennifer Taback

**ABSTRACT.** We describe a family of elements in Thompson's group  $F$  which present a challenge to finding canonical minimal length representatives for group elements, and which show that  $F$  is not combable by geodesics. These elements have the property that there are only two possible suffixes of long lengths for geodesic paths to these elements from the identity; one is of the form  $g^k$  and the other of the form  $g^{-k}$  where  $g$  is an element of a finite generating set for the group.

## 1. Introduction

Thompson's group  $F$  is a remarkable group with a poorly understood but fascinating Cayley graph  $\Gamma(F, \{x_0, x_1\})$ , using the standard finite presentation

$$F = \langle x_0, x_1 \mid [x_0 x_1^{-1}, x_0^{-1} x_1 x_0], [x_0 x_1^{-1}, x_0^{-2} x_1 x_0^2] \rangle.$$

We showed that this Cayley graph is not almost convex [6], meaning that metric balls are significantly folded in upon themselves, that there are geodesics which cannot be extended past a given point, ending in *dead end elements*, and that these dead end elements are all of depth 2 [7]. Belk and Bux show that this Cayley graph is additionally not minimally almost convex [2], and Guba uses it to show that the Dehn function of the group is quadratic [11]. Burillo [4], Guba [12] and Belk and Brown [1] have considered the growth of the group by trying to estimate the size of balls in this Cayley graph as well.

The goal of this paper is to exhibit a family of elements in  $F$  which:

- show the difficulty of constructing canonical minimal length representatives for elements of  $F$ , and
- exhibit the failure of this Cayley graph to satisfy the  $k$ -fellow traveller property for geodesics originating at the identity, and thus show that this graph is not combable by geodesics.

We approach the problem of the existence of canonical minimal length representatives for elements of  $F$  in this generating set by considering the conditions under which certain generators decrease

---

1991 *Mathematics Subject Classification.* Primary 20F65.

*Key words and phrases.* Thompson's group, geodesic combing, geometric group theory.

The first author acknowledges support from PSC-CUNY grant #64459-0033.

The second author acknowledges support from NSF grant DMS-0437481.

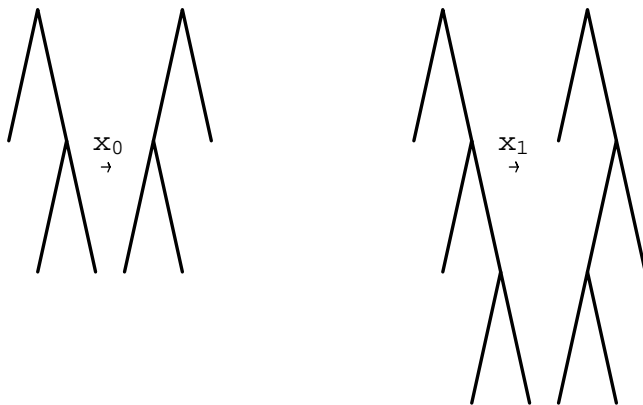


FIGURE 1. The tree pair diagrams representing the generators  $x_0$  and  $x_1$  of  $F$ .

the word length of elements. In a free group with a free generating set, there is always a unique generator which reduces the word length of a nontrivial element  $w$ . In groups with relations, understanding which generators reduce word length of given elements can give insight into the geometry of the group.

Let  $\Gamma$  be the Cayley graph of  $F$  with respect to the finite generating set  $\{x_0, x_1\}$ . All word lengths in the arguments below are computed with respect to this generating set. Our approach towards the construction of minimal length representatives for group elements is to follow Fordham's outline and begin with  $w \in F$ , then find a generator  $g \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$  so that  $|wg| = |w| - 1$ , where  $|\cdot|$  denotes word length with respect to this generating set. Iteration of this process will construct minimal length representatives, and a natural goal is to find a canonical way to proceed at elements for which several generators decrease word length.

We describe below a class of 'seesaw' words with the following property. If  $w \in F$  is a seesaw word, then there is a unique generator  $g \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$  so that  $g$  and  $g^{-1}$  are the only two generators which decrease the word length of  $w$ , that is,  $|wg^{\pm 1}| < |w|$ . Moreover, only  $g$  decreases the word length of  $wg^i$  for many iterations, and similarly only  $g^{-1}$  decreases the word length of  $wg^{-i}$  for many iterations. The existence of these words eliminates the possibility of a choice of geodesics from the identity to each elements satisfying the  $k$ -fellow traveller property.

## 2. Metric Properties of $F$

We view elements of  $F$  as pairs of finite rooted binary trees, each with the same number of leaves. To see the equivalence of this with the group presentation, we refer the reader to Cannon, Floyd and Parry [5]. We view our trees as consisting of a collection of 'carets', which are interior nodes together with their two downward-directed edges. A leaf ending in a vertex of valence one is called an *exposed leaf*. A caret may have a right child or a left child, if either or both of its leaves are not exposed. The tree pair diagrams representing the generators  $x_0$  and  $x_1$  are given in Figure 1.

There is a natural reduction condition on tree pair diagrams to ensure a unique tree pair diagram representing each group element. Namely, a tree pair diagram  $(T_-, T_+)$  is unreduced if there is a

caret with two exposed leaves, numbered  $n$  and  $n + 1$  in both trees of a tree pair diagram, and reduced otherwise. To create a reduced representative for an element, we simply remove these common carets and renumber the exposed leaves. We will assume below that if  $w = (T_-, T_+) \in F$ , then the pair of trees is reduced. We refer to  $T_-$  as the *negative tree* of the pair, and  $T_+$  as the *positive tree*.

There is an analytic interpretation of  $F$  as a group of piecewise linear homeomorphisms of the unit interval, subject to the following two conditions:

- (1) the slope of each linear piece is a power of two, and
- (2) the discontinuities of slope occur at points whose coordinates are dyadic rationals.

The carets in a binary rooted tree can be considered as instructions for dyadic subdivision of the unit interval. This gives an equivalence between tree pair diagrams and the dyadic piecewise linear homeomorphisms described above, where the trees in the tree pair diagram are used to determine the domain and range subdivisions for the homeomorphism.

With the analytic interpretation of  $F$ , group multiplication is equivalent to composition of bijective functions. In order to multiply tree pair diagrams, one mimics the condition of bijective function composition which requires that the range of one function be the domain of the other. Namely, to multiply  $w = (T_-, T_+)$  and  $v = (S_-, S_+)$ , we create unreduced representatives of the two elements,  $(T'_-, T'_+)$  and  $(S'_-, S'_+)$  respectively, in which  $T'_+ = S'_-$ . The product  $wv$  is then represented by the (not necessarily reduced) tree pair diagram  $(T'_-, S'_+)$ . See Cannon, Floyd and Parry [5] and Cleary and Taback [7] for details and examples of group multiplication of tree pair diagrams.

**2.1. Fordham's method for computing word length in  $F$ .** Viewing elements of  $F$  as tree pair diagrams, we use Fordham's method [10] for computing the word length of  $w \in F$  with respect to the standard finite generating set  $\{x_0, x_1\}$  directly from the tree pair diagram representing  $w$ . We now describe this remarkable method.

Fordham begins by dividing the carets in a binary rooted tree into distinct types, roughly left, right and interior. The left side of the tree is the path of left edges beginning at the root caret; the right side is defined analogously. Left (respectively right) carets have one leaf on the left (respectively right) side of the tree. The root caret of the tree is always considered a left caret. First, the carets are numbered using an infix numbering scheme, beginning with zero. According to this infix order, the left child of a caret is numbered before the caret, and any right child is numbered after the caret. Figure 2 provides an example of a tree pair diagram with exposed leaves numbered from left to right, and carets numbered in infix order in each tree.

Fordham's caret types are as follows:

- (1)  $L_0$ . The first caret on the left side of the tree, farthest away from the root caret. Every nonempty tree has exactly one caret of type  $L_0$ , and it will always have infix number zero.
- (2)  $L_L$ . Any other left caret.
- (3)  $I_0$ . An interior caret which has no right child.
- (4)  $I_R$ . An interior caret which has a right child.
- (5)  $R_I$ . A right caret numbered  $k$  with the property that caret  $k + 1$  is an interior caret.
- (6)  $R_0$ . A right caret with no higher-numbered interior carets.

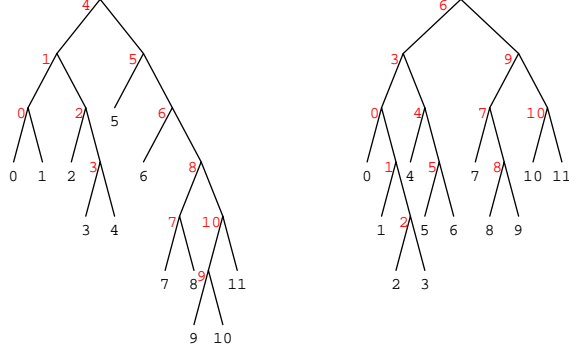


FIGURE 2. The tree pair diagram for the group element  $x_0^2 x_1 x_2 x_4 x_5 x_7 x_8 x_9^{-1} x_7^{-1} x_3^{-1} x_2^{-1} x_0^{-2}$  with caret and leaves numbered.

(7)  $R_{NI}$ . A right caret which is neither an  $R_I$  nor an  $R_0$  caret.

Let  $w \in F$  be represented by the tree pair diagram  $(T_-, T_+)$ . Fordham forms pairs of caret types by associating the carets in  $T_-$  and  $T_+$  with the same caret number; for example, the first pair consists of the type of caret zero in  $T_-$  and the type of caret zero in  $T_+$  which will necessarily be  $(L_0, L_0)$ . Each pair of caret types is assigned a weight from the following table. Notice that the pattern of weights in the table is symmetric around the diagonal. The pair  $(L_0, L_0)$  is assigned weight 0 and does not appear in the table.

	$R_0$	$R_{NI}$	$R_I$	$L_L$	$I_0$	$I_R$
$R_0$	0	2	2	1	1	3
$R_{NI}$	2	2	2	1	1	3
$R_I$	2	2	2	1	3	3
$L_L$	1	1	1	2	2	2
$I_0$	1	1	3	2	2	4
$I_R$	3	3	3	2	4	4

Fordham then proves the following theorem.

**THEOREM 2.1** (Fordham [9], Theorem 2.5.1). *Given an element  $w \in F$  described by the reduced tree pair diagram  $(T_-, T_+)$ , the word length  $|w|$  of the element with respect to the generating set  $\{x_0, x_1\}$  is the sum of the weights of the caret pairings in  $(T_-, T_+)$ .*

**2.2. Multiplication by the generators  $x_0^{\pm 1}$  and  $x_1^{\pm 1}$ .** Our method for finding minimal length representatives for elements of  $F$  with respect to the generators  $\{x_0^{\pm 1}, x_1^{\pm 1}\}$  relies on being able to recognize changes in word length from changes in caret types, using Fordham's method. Fordham proves a lemma which says that if the trees in a tree pair diagram  $w = (T_-, T_+)$  have the 'correct' shape, then multiplication by a generator will alter at most one pair of caret types in  $(T_-, T_+)$ . These conditions are seen to be necessary when one tries to perform the multiplication by creating unreduced representatives for the element  $w$  and the generator.

**LEMMA 2.1** (Fordham [10], Lemma 2.3.1). *Let  $(T_-, T_+)$  be a reduced pair of trees, each having  $m + 1$  carets, representing an element  $x \in F$ , and  $g$  a generator in  $\{x_0^{\pm 1}, x_1^{\pm 1}\}$ .*

(1) *If  $g = x_0$ , we require that the left subtree of the root of  $T_-$  is nonempty.*

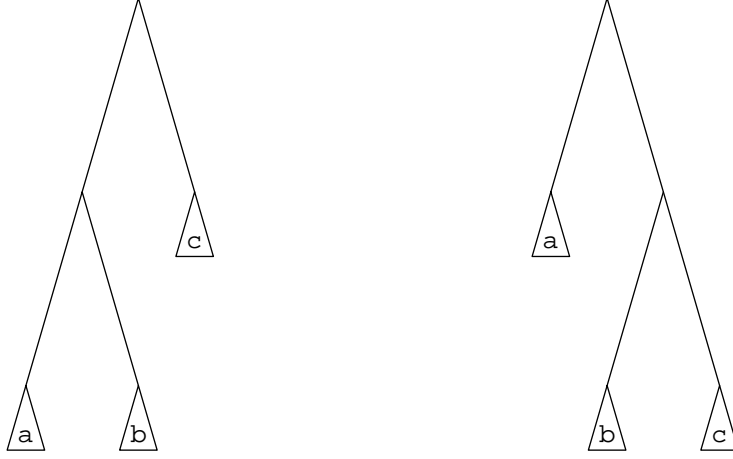


FIGURE 3. This figure depicts from left to right the negative trees from tree pair diagrams representing  $w \in F$  and  $wx_0$ . Multiplication by this generator performs a rearrangement of the subtrees of this negative tree.

- (2) If  $g = x_0^{-1}$ , we require that the right subtree of the root of  $T_-$  is nonempty.
- (3) If  $g = x_1$ , we require that the left subtree of the right child of the root of  $T_-$  is nonempty.
- (4) If  $g = x_1^{-1}$ , we require that the right subtree of the right child of the root of  $T_-$  is nonempty.

Then the reduced tree pair diagram for  $xg$  also has  $m + 1$  carets, and there is exactly one  $i$  with  $0 < i \leq m$  so that the pair of caret types of caret  $i$  changes when  $g$  is applied to  $x$ .

Furthermore, Fordham shows that if the requirements of Lemma 2.1 are not met, then there is one additional caret in the reduced tree pair diagram for  $xg$  and  $|xg| = |x| + 1$ . The elements we describe in Section 4 below will be constructed to satisfy these conditions for  $x_0$  and  $x_0^{-1}$ .

It is easy to describe the exact change in a tree pair diagram  $w = (T_-, T_+)$  under right multiplication by one of the generators  $\{x_0^{\pm 1}, x_1^{\pm 1}\}$  when  $w$  satisfies the conditions of the above lemma. Each generator induces a rearrangement of the subtrees of  $T_-$ , as can be seen either from performing multiplication directly on the tree pair diagrams, or translating back to the normal forms for elements.

LEMMA 2.2 ([6], Lemmas 2.6 and 2.7). *Let  $g \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$ , and  $w = (T_-, T_+)$  an element of  $F$  satisfying the condition of Lemma 2.1 corresponding to  $g$ . If  $wg = (S_-, S_+)$ , then  $S_+ = T_+$ , and one of the following situations applies to  $S_-$ .*

- (1) If  $g = x_0$  then  $T_-$  is given by the left tree in Figure 3 and  $S_-$  is given by the right tree in the figure.
- (2) If  $g = x_0^{-1}$  then  $T_-$  is given by the right tree in Figure 3 and  $S_-$  is given by the left tree in the figure.
- (3) If  $g = x_1$  then  $T_-$  is given by the left tree in Figure 4 and  $S_-$  is given by the right tree in the figure.
- (4) If  $g = x_1^{-1}$  then  $T_-$  is given by the right tree in Figure 4 and  $S_-$  is given by the left tree in the figure.

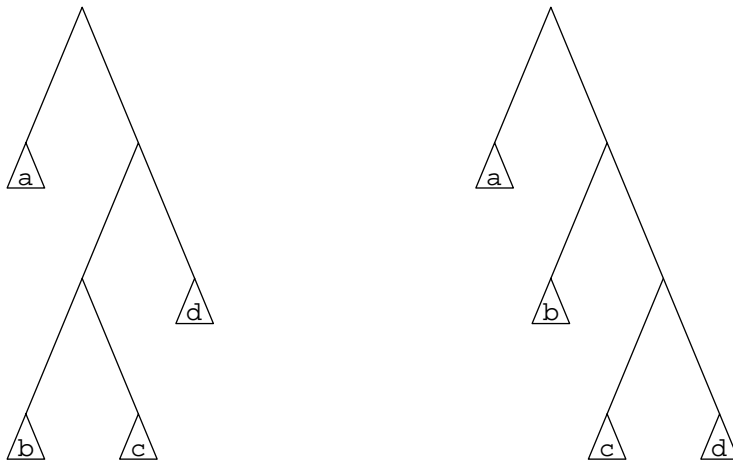


FIGURE 4. This figure depicts from left to right the negative trees from tree pair diagrams representing  $w \in F$  and  $wx_1$ . Multiplication by this generator performs a rearrangement of the subtrees of this negative tree.

Combining Lemmas 2.1 and 2.2, we see that the single caret of  $T_-$  which changes type under multiplication by a generator is in one of three possible positions: the root position, the right child of the root, or the left child of the right child of the root.

### 3. Background

**3.1. Minimal length representatives.** We now address the question of finding canonical minimal length representatives for elements  $w = (T_-, T_+) \in F$ , when word length is computed using the standard finite generating set. In [7] we present a canonical method of constructing a minimal length representative for a purely positive or purely negative word, that is, one in which either  $T_-$  or  $T_+$  is composed entirely of carets of type  $R_0$ . We call this the nested traversal method, as it creates a minimal path based on the order of the types of carets in the tree pair diagram.

The group  $F$  also has an infinite presentation, namely

$$F = \langle x_n, n \geq 0 | x_i^{-1} x_j x_i = x_{j+1} \text{ if } i < j \rangle.$$

There is a convenient set of normal forms for elements of  $F$  in this infinite presentation given by  $x_{i_1}^{r_1} x_{i_2}^{r_2} \dots x_{i_k}^{r_k} x_{j_l}^{-s_l} \dots x_{j_2}^{-s_2} x_{j_1}^{-s_1}$  with  $r_i, s_i > 0$ ,  $i_1 < i_2 < \dots < i_k$  and  $j_1 < j_2 < \dots < j_l$ . This normal form is unique if we further require that when both  $x_i$  and  $x_i^{-1}$  occur, so does  $x_{i+1}$  or  $x_{i+1}^{-1}$ , as discussed by Brown and Geoghegan in [3]. We note here that replacing each occurrence of  $x_n$  in the normal form of  $w \in F$  by  $x_0^{-n+1} x_1 x_0^{n-1}$  creates an expression for  $w$  in terms of  $\{x_0, x_1\}$  which is usually not minimal.

We attempt to create canonical minimal length representatives for elements of  $F$  by constructing a path of generators from an element  $w$  to the identity, each of which decreases word length at the given point on the path. Crucial to this are the geometric conditions detailed in Lemma 2.1.

Below we construct a family of words of  $F$  we call *seesaw* words, which have the following property. If  $w \in F$  is a seesaw word, then there is a unique generator pair  $g^{\pm 1} \in \{x_0^{\pm 1}, x_1^{\pm 1}\}$  which decreases the word length of  $w$ . Moreover, only multiplication by the generator  $g$  (resp.  $g^{-1}$ ) decreases the word length of  $wg$  (resp.  $wg^{-1}$ ) for many iterations.

**3.2. The  $k$ -fellow traveller property.** We now consider a collection of paths from the identity to each vertex in a Cayley graph. Such a collection of paths satisfies the  $k$ -fellow traveller property if paths which end at points distance one apart always stay within distance  $k$  of each other. This property is an important part of the definition of an *automatic* group, and a *combable* group has canonical paths from the identity to each element which satisfy the  $k$ -fellow traveller property for some fixed  $k$ . The elements we construct below in Section 4 show that no collection of geodesic paths in the Cayley graph  $\Gamma = \Gamma(F, \{x_0, x_1\})$  can satisfy this property.

All collections of paths we consider below in a given Cayley graph include exactly one path for each group element.

Let  $\gamma$  be a path in a Cayley graph  $\Gamma(G, X)$  from the identity to some element  $w \in G$ . Then  $\gamma = \prod_{i=1}^n g_i$  where  $g_i \in X$ . We can view  $\gamma$  as an eventually constant map from  $\mathbb{Z}^+$  into  $G$  where  $\gamma(i) = \prod_{j=1}^i g_j$  for  $i \leq n$  and  $\gamma(i) = w$  for  $i > n$ . We begin by defining the synchronous distance between two such paths.

DEFINITION 3.1. Let  $\gamma$  and  $\eta$  be paths from the identity in the Cayley graph  $\Gamma(G, X)$  to elements  $w$  and  $v$ , respectively. Then the *synchronous distance* between  $\gamma$  and  $\eta$  is defined to be

$$D_s(\gamma, \eta) = \max_{i \in \mathbb{Z}^+} d_\Gamma(\gamma(i), \eta(i)).$$

We can now define the  $k$ -fellow traveller property for a pair of paths; a collection of paths satisfies this property if every pair of paths ending at vertices one unit apart satisfies the  $k$ -fellow traveller property for the same constant  $k$ .

DEFINITION 3.2. Two paths  $\gamma$  and  $\eta$  in the Cayley graph  $\Gamma(G, X)$  from the identity to  $w$  and  $v$  respectively, with  $d_\Gamma(w, v) = 1$  are said to  *$k$ -fellow travel* if  $D_s(\gamma, \eta) \leq k$ .

#### 4. Seesaw words

We now define the seesaw words mentioned above, and show that Thompson's group  $F$  contains arbitrarily large examples of such words.

DEFINITION 4.1. A element  $w$  in a finitely generated group  $G$  with finite generating set  $X$  is a *seesaw word of swing  $k$  with respect to a generator  $g$*  if the following conditions hold. Let  $|w|$  represent the word length of  $w$  with respect to the generating set  $X$ .

- (1) Right multiplication by both  $g$  and  $g^{-1}$  reduces the word length of  $w$ ; that is,  $|wg^{\pm 1}| = |w| - 1$ , and for all  $h \in X \setminus \{g^{\pm 1}\}$ , we have  $|wh^{\pm 1}| \geq |w|$ .
- (2) Additionally,  $|wg^l| = |wg^{l-1}| - 1$  for integral  $l \in [1, k]$ , and  $|wg^m h^{\pm 1}| \geq |wg^m|$  for all  $h \in X \setminus \{g\}$  and integral  $m \in [1, k-1]$ .
- (3) Similarly,  $|wg^{-l}| = |wg^{-l+1}| - 1$  for integral  $l \in [1, k]$ , and  $|wg^{-m} h^{\pm 1}| \geq |wg^{-m}|$  for all  $h \in X \setminus \{g^{-1}\}$  for integral  $m \in [1, k-1]$ .

These are called seesaw words because they behave like a balanced seesaw. When in balance, there is a two-way choice about which way to go down, but once that initial choice is made,

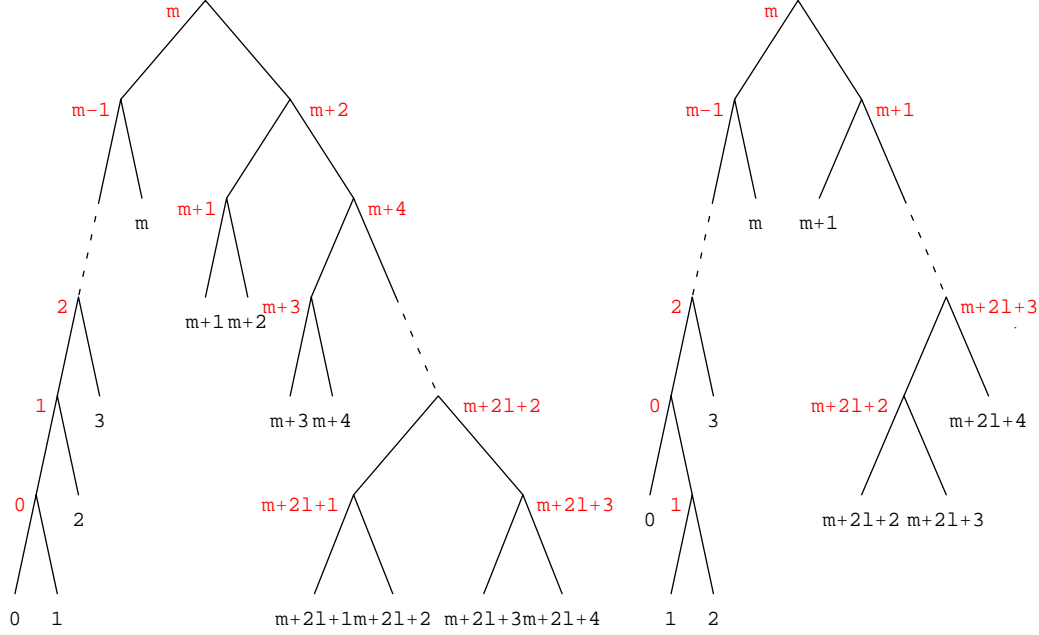


FIGURE 5. The tree pair diagram representing the seesaw word  $x_0^{m-1}x_1x_{m+2l+2}x_{m+2l+1}^{-1}x_{m+2l-1}^{-1}\dots x_{m+3}^{-1}x_{m+1}^{-1}x_0^{-m}$ . Both trees have the same number of leaves.

there is only the inexorable descent downward by the same generator for a large number of steps determined by the swing.

Finite cyclic groups  $\mathbb{Z}_{2k}$  have seesaw words of swing  $k$  with respect to the standard one-generator generating set at the point  $k$ . The only other examples of seesaw words of sizable swing known to the authors beside those described here occur in wreath products, such as  $\mathbb{Z} \wr \mathbb{Z}$  and the lamplighter groups  $\mathbb{Z}_n \wr \mathbb{Z}$ , as described in [8]. All of those wreath product examples are not finitely presentable.

**THEOREM 4.1.** *Thompson's group  $F$  contains seesaw words of arbitrarily large swing with respect to the generator  $x_0$  in the standard generating set  $\{x_0, x_1\}$ .*

**PROOF.** The idea of the proof is to construct elements  $w$  with specific pairs of caret types, chosen so that multiplication by both  $x_0$  and  $x_0^{-1}$  initially reduce the word length of  $w$ . This is easily seen using Fordham's methods. Additionally, the pairs of caret types which change under repeated multiplication by these generators are also chosen so that these generators decrease word length with each successive application. One such family of words is defined using two parameters,  $l$  and  $m$ , and these words have normal forms

$$x_0^{m-1}x_1x_{m+2l+2}x_{m+2l+1}^{-1}x_{m+2l-1}^{-1}x_{m+2l-3}^{-1}\dots x_{m+3}^{-1}x_{m+1}^{-1}x_0^{-m}.$$

An example of a seesaw word of this form is given in Figure 5. We denote the family of these words by  $\mathcal{S}$ . The parameter  $l$  in the generic word of  $\mathcal{S}$  given above determines the length of the string of  $R_I$  carets along the right side of the negative tree of the pair, and of  $R_{NI}$  carets on the



right side of the positive tree. The parameter  $m$  determines the length of the left sides of the trees. To ensure that our words in  $\mathcal{S}$  have swing at least a given  $k$ , we let  $l \geq k$  and  $m \geq k$ .

We consider what caret types the carets near the root of  $T_-$  in an element of  $\mathcal{S}$  are paired with to ensure that multiplication by both  $x_0$  and  $x_0^{-1}$  decrease word length initially. We see that for  $w = (T_-, T_+) \in \mathcal{S}$ , the root caret of  $T_-$ , numbered  $m$ , is of type  $L_L$  and is paired with caret  $m$  in  $T_+$ , also of type  $L_L$ . Caret  $m+2$ , the right child of the root in  $T_-$ , is of type  $R_I$  and is paired with caret  $m+2$  in  $T_+$ , of type  $R_{NI}$ .

We now consider which generators reduce the length of  $w$ . We first note that all our words will satisfy the conditions of Lemma 2.1, and thus the conclusions of Lemma 2.2 apply.

Right multiplication by  $x_0$  will change caret  $m$  in  $T_-$  from  $L_L$  to  $R_I$ , so the pair of caret types will change from  $(L_L, L_L)$  to  $(R_I, L_L)$ , a reduction in weight from 2 to 1 which will reduce the overall word length by 1. Right multiplication by  $x_0^{-1}$  will change caret  $m+2$  in  $T_-$  from  $R_I$  to  $L_L$ , so the pair of caret types will change from  $(R_I, R_{NI})$  to  $(L_L, R_{NI})$ , a reduction in weight from 2 to 1 which will reduce the overall word length by 1.

Right multiplication by  $x_1$  will change caret  $m+1$  in  $T_-$  from  $I_0$  to  $R_{NI}$ , so the pair of caret types will change from  $(I_0, R_{NI})$  to  $(R_{NI}, R_{NI})$ , an increase in weight from 1 to 2 which will increase the overall word length by 1.

Right multiplication by  $x_1^{-1}$  will change caret  $m+2$  in  $T_-$  from  $R_I$  to  $I_R$ , so the pair of caret types will change from  $(R_I, R_{NI})$  to  $(I_R, R_{NI})$ , an increase in weight from 2 to 3 which will increase the overall word length by 1.

Thus,  $x_0$  and  $x_0^{-1}$  reduce the word length of  $w$  while  $x_1$  and  $x_1^{-1}$  increase the word length. Now we consider how right-multiplication by each generator will affect the word length of  $w x_0^s$  for  $s$  between  $-l$  and  $m$ .

For  $1 \leq s < m$ , the root caret of the negative tree of the pair representing  $w x_0^s$  will have caret number  $m-s$  and be of type  $L_L$ , and the right child of the root will be caret number  $m-s+1$  of type  $R_{NI}$ . Both carets  $m-s$  and  $m-s+1$  will be paired with carets of type  $L_L$ . These trees satisfy the appropriate condition of Lemma 2.1. When  $w x_0^s$  is multiplied by  $x_0$ , the pair of caret types which changes corresponds to infix number  $m-s$ , and the change is from  $(L_L, L_L)$  to  $(R_{NI}, L_L)$  which will reduce length by 1. Applying  $x_0^{-1}$  will make the reverse change and increase length.

Multiplication by  $x_1^{-1}$  will change the pair of caret types of caret  $m-s+1$  either from  $(R_{NI}, R_{NI})$  to  $(I_0, R_{NI})$  or from  $(R_I, R_{NI})$  to  $(I_R, R_{NI})$ , both of which increase word length by one. The trees representing  $w x_0^s$  do not satisfy the condition of Lemma 2.1 corresponding to the generator  $x_1$ , and thus it follows that  $|w x_0^s x_1| \geq |w x_0^s|$ .

For  $-l < s \leq -1$ , the root caret of the negative tree of  $w x_0^s$  will be caret number  $m-2s$  and have caret type  $L_L$ , and the right child of the root will be caret number  $m-2s+2$  of type  $R_I$ . These will both be paired with carets of type  $R_{NI}$ , so multiplication by  $x_0^{-1}$  will change the pair of caret types corresponding to carets  $m-2s$  in both trees from  $(L_L, R_{NI})$  to  $(R_I, R_{NI})$ , which will decrease word length by 1. Multiplication by  $x_0$  will make the reverse change and increase word length by 1. Multiplication by  $x_1^{-1}$  will change the types of the carets of infix number  $m-2s+2$  from from  $(R_I, R_{NI})$  to  $(I_R, R_{NI})$  and also increase word length. Multiplication by

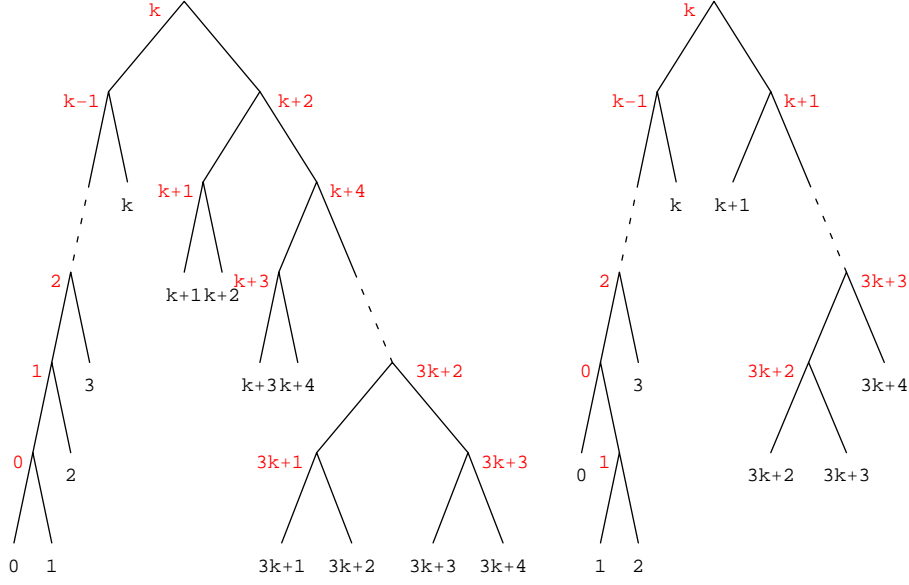


FIGURE 6. Seesaw word  $x_0^{k-1}x_1x_{3k+2}x_{3k+1}^{-1}x_{3k-1}^{-1}\dots x_{k+3}^{-1}x_{k+1}^{-1}x_0^{-k}$  of swing  $k$ .

$x_1$  will change the types of the caret of infix number  $m - 2s + 1$  from  $(I_0, R_{NI})$  to  $(R_{NI}, R_{NI})$ , increasing the word length by one.

Thus we see that all  $w \in \mathcal{S}$  are seesaw words, and that there are such words of any swing  $k$ .  $\square$

The seesaw words used in the proof of Theorem 4.1 are potentially asymmetric, in that the two parameters  $l$  and  $m$  separately control the extent to which  $x_0^{-1}$  and  $x_0$  respectively reduce word length. For simplicity, we can consider a one-parameter family of seesaw words where we set  $l = m = k$  to get seesaw words of swing  $k$  in both directions. These words are of the form  $x_0^{k-1}x_1x_{3k+2}x_{3k+1}^{-1}x_{3k-1}^{-1}\dots x_{k+3}^{-1}x_{k+1}^{-1}x_0^{-k}$  and are pictured in Figure 6.

The existence of these seesaw words eliminates the possibility of families of geodesics in the Cayley graph  $\Gamma(F, \{x_0, x_1\})$  which satisfy the  $k$ -fellow traveller property.

**PROPOSITION 4.2.** *Given any constant  $k$ , there is  $w \in \mathcal{S}$  so that  $w, wx_0$  and  $wx_0^{-1}$  cannot be represented by geodesic paths from the identity which satisfy the  $k$ -fellow traveller property.*

**PROOF.** Let  $w \in \mathcal{S}$  be a seesaw word of swing  $m$ . Then any geodesic path from the identity to  $w$  in the Cayley graph  $\Gamma$  must end either in the suffix  $x_0^m$  or  $x_0^{-m}$ . The first of these possible suffixes comes from a path which passes through  $wx_0^{-1}$  and the second from a path through  $wx_0$ .

Let  $\gamma$  be a geodesic path from the identity to  $w$  passing through  $wx_0$  and  $\eta'$  a path from the identity to  $w$  passing through  $wx_0^{-1}$ . Let  $\eta$  be the prefix of this path ending at  $wx_0^{-1}$ . Then the length of  $\gamma$  is one more than the length of  $\eta$ , and  $d_\Gamma(w, wx_0^{-1}) = 1$ .

We can write  $\gamma = \gamma_1 x_0^s$  and  $\eta = \eta_1 x_0^{-(s-1)}$ , where  $s \leq m$ . Then we know that  $\gamma_1$  and  $\eta_1$  have the same length. To compute the distance  $D_s(\gamma, \eta)$ , we consider  $d(wx_0^{-s}, wx_0^s) = 2s$ . Since we

can find seesaw words of arbitrarily large swing, we can make this distance arbitrarily large. Thus  $D_s(\gamma, \eta)$  is not bounded by a constant, and the paths cannot satisfy the  $k$ -fellow traveller property for the given constant  $k$ .  $\square$

The paths discussed above show explicitly that  $F$  does not admit a combing by geodesics. This also follows from the fact that  $F$  is not almost convex [6].

**THEOREM 4.2.** *Thompson's group  $F$  is not combable by geodesics.*

**PROOF.** Consider any combing of the Cayley graph  $\Gamma$  by geodesics. Let  $w \in \mathcal{S}$ , and  $\gamma$  the geodesic combing path from the identity to  $w$  in  $\Gamma$ . Then  $\gamma$  passes through  $wx_0$  or  $wx_0^{-1}$  but not both. Let  $\eta$  be the combing path to the point  $wx_0^{\pm 1}$  not on  $\gamma$ . It follows from Proposition 4.2 that these paths do not satisfy the  $k$  fellow traveller property, and thus  $\Gamma$  is not combable by geodesics.  $\square$

## References

- [1] James Belk and Kenneth S. Brown. Forest diagrams for elements of Thompson's group  $F$ , arXiv:math.GR/0305412. Preprint.
- [2] James Belk and Kai-Uwe Bux. Thompson's group  $F$  is not minimally almost convex, arXiv:math.GR/0301141. Preprint.
- [3] Kenneth S. Brown and Ross Geoghegan. An infinite-dimensional torsion-free  $FP_\infty$  group. *Inventiones mathematicae*, 77:367–381, 1984.
- [4] José Burillo. Growth of positive words in Thompson's group  $F$ . Preprint.
- [5] James W. Cannon, William J. Floyd, and Walter R. Parry. Introductory notes on Richard Thompson's groups. *L'Ens. Math.*, 42:215–256, 1996.
- [6] Sean Cleary and Jennifer Taback. Thompson's group  $F$  is not almost convex. *J. Algebra*, 270(1):133–149, 2003.
- [7] Sean Cleary and Jennifer Taback. Combinatorial properties of Thompson's group  $F$ . *Trans. Amer. Math. Soc.*, 356(7):2825–2849 (electronic), 2004.
- [8] Sean Cleary and Jennifer Taback. Dead end words in lamplighter groups and other wreath products. *Quarterly Journal of Mathematics*, to appear, arXiv:math.GR/0309344.
- [9] Blake Fordham. *Minimal Length Elements of Thompson's group  $F$* . PhD thesis, Brigham Young Univ, 1995.
- [10] S. Blake Fordham. Minimal length elements of Thompson's group  $F$ . *Geom. Dedicata*, 99:179–220, 2003.
- [11] Victor Guba. The Dehn function of Thompson's group  $f$  is quadratic, arXiv:math.GR/0211395. Preprint.
- [12] Victor Guba. On the properties of the Cayley graph of Richard Thompson's group  $F$ , arXiv:math.GR/0211396. Preprint.

DEPARTMENT OF MATHEMATICS, THE CITY COLLEGE OF NEW YORK, NEW YORK, NY 10031

*E-mail address:* cleary@sci.ccny.cuny.edu

DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE, BRUNSWICK, ME 04011

*E-mail address:* jtaback@bowdoin.edu